SOMEWHERE DENSE ORBIT OF ABELIAN SUBGROUP OF DIFFEOMORPHISMS MAPS ACTING ON $\mathbb{C}^n$

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ABSTRACT. In this paper, we give a characterization for any abelian subgroup G of a lie group of diffeomorphisms maps of $\mathbb{C}^n$, having a somewhere dense orbit $G(x)$, $x \in \mathbb{C}^n$: $G(x)$ is somewhere dense in $\mathbb{C}^n$ if and only if there are $f_1, \dots, f_{2n+1} \in exp^{-1}(G)$ such that $f_{2n+1} \in vect(f_1, \dots, f_{2n})$ and $\mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)$ is dense in $\mathbb{C}^n$, where $vect(f_1, \dots, f_{2n})$ is the vector space over $\mathbb{R}$ generated by $f_1, \dots, f_{2n}$.

1. Introduction

Denote by $Diff^r(\mathbb{C}^n)$, $r \geq 1$ the group of all C^r-diffemorphisms of $\mathbb{C}^n$. Let Γ be a lie subgroup of $Diff^r(\mathbb{C}^n)$, $r \geq 1$ and G be an abelian subgroup of Γ, such that $Fix(G) \neq \emptyset$, where $Fix(G) = \{x \in \mathbb{C}^n : \ f(x) = x, \ \forall f \in G\}$ be the global fixed point set of G. There is a natural action $G \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$. $(f, x) \longmapsto f(x)$. For a point $x \in \mathbb{C}^n$, denote by $G(x) = \{f(x), \ f \in G\} \subset \mathbb{C}^n$ the orbit of G through x. A subset $E \subset \mathbb{C}^n$ is called G-invariant if $f(E) \subset E$ for any $f \in G$; that is E is a union of orbits. Denote by $\overline{E}$ (resp. $\overset{\circ}{E}$) the closure (resp. interior) of E.

Recall that $E \subset \mathbb{C}^n$ is somewhere dense in $\mathbb{C}^n$ if the closure $\overline{E}$ has nonempty interior in $\mathbb{C}^n$. An orbit γ is called somewhere dense (or locally dense) if $\overset{\circ}{\overline{\gamma}} \neq \emptyset$. The group G is called hypercyclic if it has a dense orbit in $\mathbb{C}^n$. Hypercyclic is also called topologically transitive.

The purpose of this paper is to give a characterization for any subgroup G of a lie group of diffeomorphisms maps of $\mathbb{C}^n$, having a dense orbit. In [1], the authors present a global dynamic of every abelian subgroup of $GL(n, \mathbb{C})$ and in [2], they characterize hypercyclic abelian subgroup of $GL(n, \mathbb{C})$. Our main result is viewed as a continuation of [7] and [8].
Denote by:
- $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$.
- $C^r(\mathbb{C}^n, \mathbb{C}^n)$ the set of all C^r-differentiable maps of $\mathbb{C}^n$.
- For a subset $E \subset \mathbb{C}^n$ (resp. $E \subset C^r(\mathbb{C}^n, \mathbb{C}^n)$), denote by $vect(E)$ the vector subspace of $\mathbb{C}^n$ (resp. $C^r(\mathbb{C}^n, \mathbb{C}^n)$) over $\mathbb{R}$ generated by all elements of E.
- $exp : \ C^r(\mathbb{C}^n, \mathbb{C}^n) \longrightarrow Diff^r(\mathbb{C}^n)$ the exponential map defined by $exp(f) = e^f$, $f \in C^r(\mathbb{C}^n, \mathbb{C}^n)$.

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- H the lie algebra associated to Γ.
- $exp : H \longrightarrow \Gamma$ be the exponential map.
- $H_x = \{f(x), \quad B \in H\}$, it is a vector subspace of $\mathbb{C}^n$ over $\mathbb{R}$.
- $\mathrm{g} = exp^{-1}(G)$, it is an additive group because G is abelian.
- $\mathrm{g}_x = \{f(x), \quad B \in \mathrm{g}\}$, it is an additive subgroup of $\mathbb{C}^n$ because g is an additive group.

Our principal results can be stated as follows:

Theorem 1.1. *Let Γ be an abelian lie subgroup of $Diff^r(\mathbb{C}^n)$ and $x \in \mathbb{C}^n \backslash \{0\}$. Then the following assertions are equivalent:*
(i) $H_x = \mathbb{C}^n$.
(ii) $\overset{\circ}{\overline{\Gamma(x)}} \neq \emptyset$.

In general, the Lie algebra $\widetilde{\mathrm{g}}$ is not explicitly defined, so we give an explicitly test to the existence of somewhere dense orbit by the following theorem:

Theorem 1.2. *Let G be an abelian subgroup of a lie group $\Gamma \subset Diff^r(\mathbb{C}^n)$ and $x \in \mathbb{C}^n \backslash \{0\}$. Then $\overset{\circ}{\overline{G(x)}} \neq \emptyset$ if and only if there exist $f_1, \dots, f_{2n+1} \in exp^{-1}(\widetilde{G})$ such that $f_{2n+1} \in vect(f_1, \dots, f_{2n})$ and $\mathbb{Z}f_1(x) + \cdots + \mathbb{Z}f_{2n+1}(x)$ is a dense additive subgroup of $\mathbb{C}^n$.*

Let's introduce the arithmetic property: We say that $f_1, \dots, f_{2n+1} \in C^r(\mathbb{C}^n, \mathbb{C}^n)$ satisfy *property $\mathcal{D}(x)$* for some $x \in \mathbb{C}^n$ if $f_1, \dots, f_{2n}$ are linearly independent, $f_{2n+1} \in vect(f_1, \dots, f_{2n})$ and for every $(s_1, \dots, s_{2n+1}) \in \mathbb{Z}^{2n+1} \backslash \{0\}$:

$$\mathrm{rank} \begin{bmatrix} \mathrm{Re}(f_1(x)) & \dots & \mathrm{Re}(f_{2n+1}(x)) \\ \mathrm{Im}(f_1(x)) & \dots & \mathrm{Im}(f_{2n+1}(x)) \\ s_1 & \dots & s_{2n+1} \end{bmatrix} = 2n + 1.$$

For a vector $v \in \mathbb{C}^n$, we write $v = \mathrm{Re}(v) + i\mathrm{Im}(v)$ where $\mathrm{Re}(v)$ and $\mathrm{Im}(v) \in \mathbb{R}^n$.

As an immediate consequence of Theorem 1.2, we have:

Corollary 1.3. *Let G be an abelian subgroup of a lie group $\Gamma \subset Diff^r(\mathbb{C}^n)$ and $x \in \mathbb{C}^n \backslash \{0\}$. Then $\overset{\circ}{\overline{G(x)}} \neq \emptyset$ if and only if there exist $f_1, \dots, f_{2n+1} \in exp^{-1}(G)$ and satisfying property $\mathcal{D}(x)$.*

As an important consequence of the Theorem 1.2, we give the following Corollary which simplifies the test given by Theorem 1.3 proved in [2] for the abelian subgroup of $GL(n, \mathbb{C})$:

Corollary 1.4. *Let G be an abelian subgroup of $GL(n, \mathbb{C})$ and $x \in \mathbb{C}^n \backslash \{0\}$. Then $\overline{G(x)} = \mathbb{C}^n$ if and only if there exist $B_1, \dots, B_{2n+1} \in exp^{-1}(G)$ such that $\mathbb{Z}B_1 x + \cdots + \mathbb{Z}B_{2n+1} x$ is dense in $\mathbb{C}^n$.*

This paper is organized as follows: In Section 2 we prove Theorem 1.1. Section3 is devoted to prove Theorem 1.2 and Corollaries 1.3, 1.4.

2. **Proof of Theorem 1.1**

We will cite the definition of the exponential map given in [3].

2.1. **Exponential map.** In this section, we illustrate the theory developed of the group $Diff(\mathbb{C}^n)$ of diffeomorphisms of $\mathbb{C}^n$. For simplicity, throughout this section we only consider the case of $\mathbb{C} = \mathbb{R}$; however, all results also hold for complexes case. The group $Diff(\mathbb{R}^n)$ is not a Lie group (it is infinite-dimensional), but in many way it is similar to Lie groups. For example, it easy to define what a smooth map from some Lie group G to $Diff(\mathbb{R}^n)$ is: it is the same as an action of G on $\mathbb{R}^n$ by diffeomorphisms. Ignoring the technical problem with infinite-dimensionality for now, let us try to see what is the natural analog of the Lie algebra g for the group G. It should be the tangent space at the identity; thus, its elements are derivatives of one-parameter families of diffeomorphisms.
Let $\varphi^t : G \longrightarrow G$ be one-parameter family of diffeomorphisms. Then, for every point $a \in G$, $\varphi^t(a)$ is a curve in G and thus $\frac{\partial}{\partial t}\varphi^t(a)_{/t=0} = \xi(a) \in T_aG$ is a tangent vector to G at m. In other words, $\frac{\partial}{\partial t}\varphi^t$ is a vector field on G.

The exponential map $exp : \mathrm{g} \longrightarrow G$ is defined by $exp(x) = \gamma_x(1)$ where $\gamma_x(t)$ is the one-parameter subgroup with tangent vector at 1 equal to x.
If $\xi \in \mathrm{g}$ is a vectorfield, then $exp(t\xi)$ should be one-parameter family of diffeomorphisms whose derivative is vector field ξ. So this is the solution of differential equation

$$\frac{\partial}{\partial t}\varphi^t(a)_{/t=0} = \xi(a).$$

In other words, φ^t is the time t flow of the vector field. Thus, it is natural to define the Lie algebra of G to be the space g of all smooth vector ξ fields on $\mathbb{R}^n$ such that $exp(t\xi) \in G$ for every $t \in \mathbb{R}$.

We will use the definition of Whitney topology given in [6].

2.2. **Whitney Topology on $C^0(\mathbb{C}^n, \mathbb{C}^n)$.** For each open subset $U \subset \mathbb{C}^n \times \mathbb{C}^n$ let $\widetilde{U} \subset \mathcal{C}^0(\mathbb{C}^n, \mathbb{C}^n)$ be the set of continuous functions g, whose graphs $\{(x, g(x)) \in \mathbb{C}^n \times \mathbb{C}^n,\ \ x \in \mathbb{C}^n\}$ is contained in U. We want to construct a neighborhood basis of each function $f \in \mathcal{C}^0(\mathbb{C}^n, \mathbb{C}^n)$. Let $K_j = \{x \in \mathbb{C}^n,\ \ \|x\| \leq j\}$ be a countable family of compact sets (closed balls with center 0) covering $\mathbb{C}^n$ such that K_j is contained in the interior of K_{j+1}. Consider then the compact subsets $L_j = K_j \backslash \overbrace{K_{j-1}}^{\circ}$, which are compact sets, too. Let $\epsilon = (\varepsilon_j)_j$ be a sequence of positive numbers and then define

$$V_{(f;\epsilon)} = \{f \in \mathcal{C}^0(\mathbb{C}^n, \mathbb{C}^n)\ :\ \|f(x) - g(x)\| < \varepsilon_j,\ \text{for any } x \in L_j,\ \forall j\}.$$

We claim this is a neighborhood system of the function f in $\mathcal{C}^0(\mathbb{C}^n, \mathbb{C}^n)$. Since L_i is compact, the set $U = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n\ :\ \|f(x) - g(x)\| < \varepsilon_j,\ if\ x \in L_j\}$ is open. Thus, $V_{(f;\epsilon)} = \widetilde{U}$ is an open neighborhood of f. On the other hand, if O is an open subset of $\mathbb{C}^n \times \mathbb{C}^n$ which contains the graph of f, then since L_j is compact, it follows that there exists $\varepsilon_j > 0$ such that if $x \in L_j$ and $\|y - f(x)\| < \varepsilon_j$, then $(x; y) \in O$. Thus, taking $\widetilde{\epsilon} = (\varepsilon_j)_j$ we have $V_{(f;\widetilde{\epsilon})} \subset \widetilde{O}$, so we have obtained the family $V_{(f;\epsilon)}$ is a neighborhood system of f. Moreover, for each given $\epsilon = (\varepsilon_j)_j$,

we can find a C^∞-function $\epsilon : \mathbb{C}^n \longrightarrow \mathbb{R}_+$, such that $\epsilon(x) < \varepsilon_j$ for any $x \in L_j$. It follows that the family $V_{(f;\epsilon)} = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \ : \ \|f(x) - g(x)\| < \epsilon(x)\}$ is also a neighborhood system.

Denote by:
- $\widetilde{G} = \overline{G} \cap Diff^r(\mathbb{C}^n)$, where $\overline{G}$ is the closure of G in $C^r(\mathbb{C}^n, \mathbb{C}^n)$ for the withney topology defined above. So $\widetilde{G}$ is an abelian lie subgroup of Γ.
- $\mathcal{A}(\widetilde{G})$ the algebra generated by G. See that $G \subset \mathcal{A}(\widetilde{G})$.
- $\Phi_x : \mathcal{A}(\widetilde{G}) \longrightarrow \mathbb{C}^n$ the linear map given by $\Phi_x(f) = f(x)$, $f \in \mathcal{A}(\widetilde{G})$.
- $E(x) = \Phi_x(\mathcal{A}(G))$.

Lemma 2.1. *The linear map $\Phi_x : \mathcal{A}(\widetilde{G}) \longrightarrow E(x)$ is continuous.*

Proof. Firstly, we take the restriction of the Whitney topology to $\mathcal{A}(\widetilde{G})$. Secondly, let $f \in \mathcal{A}(\widetilde{G})$ and $\varepsilon > 0$. Then for $\epsilon = (\varepsilon_j)_j$ with $\varepsilon_j = \varepsilon$ and for $V_{(f;\epsilon)}$ be a neighborhood system of f, we obtain: for every $g \in V_{(f;\epsilon)} \cap \mathcal{A}(\widetilde{G})$ and for every $y \in L_j$, $\|f(y) - g(y)\| < \varepsilon$, $\forall j$. In particular for $j = j_0$ in which $x \in L_{j_0}$, we have $\|f(x) - g(x)\| < \varepsilon$, so $\|\Phi_x(f) - \Phi_x(g)\| < \varepsilon$. It follows that Φ_x is continuous. $\square$

2.3. **Proof of Theorem 1.1.**

Proposition 2.2. ([3], *Theorem 3.29*) *Let G be a Lie group acting on $\mathbb{C}^n$ with lie algebra $\widetilde{\mathrm{g}}$ and let $u \in \mathbb{C}^n$.*
(i) The stabilizer $G_x = \{B \in G : \ \ Bu = u\}$ is a closed Lie subgroup in G, with Lie algebra $\mathfrak{h}_x = \{B \in \widetilde{\mathrm{g}} : \ Bu = 0\}$.
(ii) The map $G_{/G_x} \longrightarrow \mathbb{C}^n$ given by $B.G_x \longmapsto Bu$ is an immersion. Thus, the orbit $G(x)$ is an immersed submanifold in $\mathbb{C}^n$. In particular $\dim(G(x)) = \dim(\widetilde{\mathrm{g}}) - \dim(\mathfrak{h}_x)$.

Here $\mathfrak{h}_x = Ker(\Phi_x)$ since $Ker(\Phi_x) \subset \widetilde{\mathrm{g}}$. Write:
- $\widetilde{L}$ the vector subspace of $\widetilde{\mathrm{g}}$ supplement to $Ker(\Phi_x)$, (i.e. $\widetilde{L} \oplus Ker(\Phi_x) = \widetilde{\mathrm{g}}$). It is clear that $\dim(\widetilde{L}) = cod(Ker(\Phi_x)) \leq n$, then $\widetilde{L}$ is closed.
- $exp : \widetilde{L} \oplus Ker(\Phi_x) \longrightarrow \widetilde{G}$ the exponential map. Since $\widetilde{G}$ is abelian, so is $\widetilde{\mathrm{g}}$, then $exp(f + h) = exp(f) \circ exp(h)$ for every $f \in \widetilde{L}$ and $h \in Ker(\Phi_x)$.
- $\widetilde{G}_x$ the stabilizer of $\widetilde{G}$ on the point u. So it is a lie subgroup of $\widetilde{G}$ with lie algebra $Ker(\Phi_x)$.

As a directly consequence of Proposition 5.13, given in [4], applied to Γ, we have the following Lemma:

Lemma 2.3. ([4], *Proposition 5.13*) *Let G be an abelian subgroup of a lie group Γ. There exists an open neighborhood U of 0 in H such that $exp : U \longrightarrow exp(U)$ is a diffeomorphism and $exp(U \cap \widetilde{\mathrm{g}}) = exp(U) \cap \widetilde{G}$.*

Denote by $V = exp(U)$, where U is the open set defined in Lemma 2.3.

Lemma 2.4. *We have $\overline{G(x)} = \overline{\widetilde{G}(x)}$.*

Proof. It is clear that $\overline{G(x)} \subset \overline{\widetilde{G}(x)} \subset \overline{\overline{G}(x)}$. Let $v \in \overline{\overline{G}(x)}$, so $v = \underset{m\to+\infty}{lim} f_m(x)$ for some sequence $(f_m)_{m\in\mathbb{N}}$ in $\overline{G}$. Then for every $m \in \mathbb{N}$, there exists a sequence $(f_{m,k})_{k\in\mathbb{N}}$ in G such that $\underset{k\to+\infty}{lim} f_{m,k} = f_m$, so by continuity of Φ_x (Lemma 2.1), we have $\underset{k\to+\infty}{lim} f_{m,k}(x) = f_m(x)$, thus for every $\varepsilon > 0$, there exists $M > 0$ and for every $m \geq M$, there exists $k_m > 0$, such that $\|f_m(x) - v\| < \frac{\varepsilon}{2}$ and for every $k \geq k_m$, $\|f_{m,k}(x) - f_m(x)\| < \frac{\varepsilon}{2}$. Then, for every $m > M$,

$$\|f_{m,k_m}(x) - v\| \leq \|f_{m,k_m}(x) - f_m(x)\| + \|f_m(x) - v\| < \varepsilon,$$

therefore $\underset{m\to+\infty}{lim} f_{m,k_m}(x) = v$. Hence $v \in \overline{G(x)}$. It follows that $\overline{\widetilde{G}(x)} \subset \overline{\overline{G}(x)} \subset \overline{G(x)}$. □

Lemma 2.5. *Let $W = \Phi_x(V)$. Then $\Phi_x^{-1}(\widetilde{G}(x) \cap W) \cap V = \widetilde{G} \cap V$.*

Proof. Since $W = \Phi_x(V)$, it is obvious that $\widetilde{G} \cap V \subset \Phi_x^{-1}(\widetilde{G}(x) \cap W) \cap V$. Let $f \in \Phi_x^{-1}(\widetilde{G}(x) \cap W)$. Then there exists $g \in \widetilde{G} \cap V$ such that $f(x) = g(x)$. So $g^{-1} \circ f(x) = x$. Hence $g^{-1} \circ f \in H_x$, where H_x be the lie group generated by $\{h \in Diff^r(\mathbb{C}^n) : \ h(x) = x\} \cap \mathcal{A}(\widetilde{G})$. So H_x is contained in the stabilizer of $Diff^r(\mathbb{C}^n)$ on x. Set L_x be the lie algebra of H_x, so $L_x \subset \{h \in Diff^r(\mathbb{C}^n) : \ h(x) = 0\} \cap \mathcal{A}(\widetilde{G})$. Therefore $L_x \subset Ker(\Phi_x) \subset \widetilde{\mathrm{g}}$. Hence $H_x \subset \widetilde{G}$. It follows that $g^{-1} \circ f \in \widetilde{G}$, so $f \in \widetilde{G} \cap V$. This completes he proof. □

Proof of Theorem 1.1.
Since $\widetilde{G}$ is a locally closed sub-manifold of $Diff^r(\mathbb{C}^n)$. By Proposition 2.2.(ii), $\widetilde{G}(x)$ is an immersed submanifold of $\mathbb{C}^n$ with dimension $r = \dim(\widetilde{\mathrm{g}}) - \dim(Ker(\Phi_x))$. We have $\mathrm{Im}(\Phi_x) = \widetilde{\mathrm{g}}_x$. Then $\dim(\widetilde{\mathrm{g}}_x) = \dim(\widetilde{\mathrm{g}}) - \dim(Ker(\Phi_x))$. It follows from Proposition 2.2,(ii) that

$$\dim(\widetilde{G}(x)) = \dim(\widetilde{\mathrm{g}}_x) \quad (2)$$

.
Proof of (i) $\Longrightarrow$ (iii) : The proof results directly from (2), and the fact that $dim(\widetilde{G}(x)) = n$ if and only if $\widetilde{G}(x)$ is a non empty open set.
Proof of (iii) $\Longrightarrow$ (ii) : Since $\widetilde{G}(x) \cap W$ is a non empty open set then the proof follows directly from Lemma 2.4.(ii), because $\widetilde{G}(x) \cap W \subset \overline{\widetilde{G}(x)} \cap W = \overline{G(x)} \cap W$.
Proof of (ii) $\Longrightarrow$ (i) : Since $\overset{\circ}{\overline{G(x)}} \subset Im(\Phi_x) \subset \mathbb{C}^n$ then the linear map $\Phi_x : \mathcal{A}(\widetilde{G}) \longrightarrow \mathbb{C}^n$ is surjective, so it is an open map. By Lemma 2.3 there exists two open subsets U and $V = exp(U)$ respectively of H and Γ such that the exponential map $exp : U \longrightarrow V$ is a diffeomorphism and satisfying $exp(\widetilde{\mathrm{g}} \cap U) = \widetilde{G} \cap V$. So

$$exp^{-1}(\widetilde{G} \cap V) = \widetilde{\mathrm{g}} \cap U. \qquad (1)$$

Recall that $W = \Phi_x(V)$. Since Φ_x is an open map and by Lemma 2.4.(i), $\overset{\circ}{\overline{G(x)}} = \overset{\circ}{\overline{\widetilde{G}(x)}}$, so

$$\begin{aligned}\Phi_x^{-1}(\overset{\circ}{\overline{G(x)}} \cap W) &= \Phi_x^{-1}(\overset{\circ}{\overline{\widetilde{G}(x)}} \cap W)\\ &\subset \Phi_x^{-1}(\overline{\widetilde{G}(x) \cap W})\\ &\subset \overline{\Phi_x^{-1}(\widetilde{G}(x) \cap W)} \qquad (3)\end{aligned}$$

We have

$$\begin{aligned}\Phi_x \circ\ exp^{-1}(\Phi_x^{-1}(\overset{\circ}{\overline{G(x)}} \cap W) \cap V) &\subset \Phi_x \circ\ exp^{-1}(\overline{\Phi_x^{-1}(\widetilde{G}(x) \cap W) \cap V}) \qquad &\text{(by (3))}\\ &\subset \Phi_x \circ\ exp^{-1}(\overline{\widetilde{G} \cap V}), &\text{(by Lemma 2.5)}\\ &\subset \Phi_x \circ\ \overline{exp^{-1}(\widetilde{G} \cap V)}\\ &\subset \Phi_x(\overline{\widetilde{\mathrm{g}} \cap U}) &\text{(by (1))}\\ &\subset \widetilde{\mathrm{g}}_x\end{aligned}$$

Since $\overset{\circ}{\overline{G(x)}} \cap W$ is a non empty open subset of $\mathbb{C}^n$ then $\Phi_x \circ exp^{-1}(\Phi_x^{-1}(\overset{\circ}{\overline{G(x)}} \cap W) \cap V)$ is an open subset of $\mathbb{C}^n$. It follows that $\widetilde{\mathrm{g}}_x = \mathbb{C}^n$. The proof is completed $\square$.

3. Proof of Theorem 1.2 and Corollaries 1.3, 1.4

Under the notation of Lemma 2.3, recall that there exists an open subset U of $\mathcal{A}(\widetilde{G})$ such that $exp : U \longrightarrow exp(U)$ is a diffeomorphism. Now, by using the restricton of the withney topology to $\mathcal{A}(\widetilde{G})$, denote by:
- $B_{(0,r)} = \{f \in \mathcal{A}(\widetilde{G}) : \|f\| < r\}$, the open ball with center 0 and radius $r > 0$.
- $r_G = sup\{r \in]0,1[: B_{(0,r)} \subset U\}$, it is dependent of G since is U.

Theorem 3.1. *Let G be a subgroup of $Diff^r(\mathbb{C}^n)$ and $x \in \mathbb{C}^n$. If there exist $f_1, \dots, f_{2n} \in exp^{-1}(\widetilde{G})$ with $\|f_k\| < r_{\widetilde{G}}$, for every $k = 1, \dots, 2n$ such that $(f_1(x), \dots, f_{2n}(x))$ is a basis of $\mathbb{C}^n$ over $\mathbb{R}$, then $\overset{\circ}{\overline{G(x)}} \neq \emptyset$.*

Proof. We have $f_k \in exp^{-1}(\widetilde{G})$ with $\|f_k\| < r_{\widetilde{G}}$ for every $k = 1, \dots, 2n$, then $f_1, \dots, f_{2n} \in U$ and so $e^{f_k} \in \widetilde{G} \cap V$. By Lemma 2.3, $\widetilde{G} \cap V = exp(U \cap \widetilde{\mathrm{g}})$, hence $f_k \in \widetilde{\mathrm{g}}$, for every $k = 1, \dots, 2n$. As $(f_1(x), \dots, f_{2n}(x))$ is a basis of $\mathbb{C}^n$ over $\mathbb{R}$ then $\widetilde{\mathrm{g}_x} = \mathbb{C}^n$. It follows by Theorem 1.1 that $\overset{\circ}{\overline{G(x)}} \neq \emptyset$. $\square$

Lemma 3.2. *Let H be a vector space with dimension $2n$ over $\mathbb{R}$ and $x_1, \dots, x_{2n+1} \in H$, such that $\mathbb{Z}x_1 + \dots + \mathbb{Z}x_{2n+1}$ is dense in H. Then for every $1 \leq k \leq 2n+1$, $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{2n+1})$ is a basis of H over $\mathbb{R}$.*

Proof. We have H is isomorphic to $\mathbb{C}^n$. Let $1 \leq k \leq 2n+1$ be a fixed integer and take

$$K = vect(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{2n+1}).$$

Suppose that $\dim(K) = p < 2n$. Let $(x_{k_1}, \dots, x_{k_p})$ be a basis of K. Then $\mathbb{Z}x_1 + \dots + \mathbb{Z}x_{2n+1} \subset K + \mathbb{Z}x_k$ which cannot be dense in H, a contradiction. $\square$

Recall that $\widetilde{L}$ is the vector subspace of $\widetilde{\mathrm{g}}$ supplement to $Ker(\Phi_x)$, (i.e. $\widetilde{L} \oplus Ker(\Phi_x) = \widetilde{\mathrm{g}}$). Denote by:
- $p_x : \widetilde{L} \oplus Ker(\Phi_x) \longrightarrow \widetilde{L}$ given by $p_x(f+h) = f$, $f \in \widetilde{L}$ and $h \in Ker(\Phi_x)$.

Lemma 3.3. *Under above notations, we have:*

(i) *The linear map $\Phi_x : \widetilde{L} \longrightarrow E(x)$ defined by $\Phi_x(f) = f(x)$, is an isomorphism.*
(ii) *for every $f \in \widetilde{\mathrm{g}}$ one has $\Phi_x^{-1}(f(x)) = p_x(f)$.*

Proof. (i) By construction Φ_x is surjective and restreint to $\widetilde{L}$ it became injective. By Lemma 2.1 Φ_x is continuous and bijective. Hence it is an isomorphism because it is linear.
(ii) Let $f \in \widetilde{\mathrm{g}}$. Write $f = f_1 + f_0$ with $f_1 = p_x(f) \in \widetilde{L}$ and $f_0 \in Ker(\Phi_x)$). Since $f_0(x) = 0$, so $f(x) = f_1(x)$. By (i), Φ_x is an isomorphism, then $\Phi_x^{-1}(f(x)) = \Phi_x^{-1}(f_1(x)) = f_1 = p_x(f)$. This completes the proof. $\square$

Let $f_1, \dots, f_{2n+1} \in \widetilde{\mathrm{g}}$ and suppose that $(p_x(f_1), \dots, p_x(f_{2n}))$ is a basis of $\widetilde{L}$ over $\mathbb{R}$ and $f_{n+1} \in vect(f_1, \dots, f_{2n})$. Denote by $\Psi : \widetilde{L} \longrightarrow \widetilde{\mathrm{g}}$ the linear map given by

$$\Psi\left(\sum_{k=1}^{2n} \alpha_k p_x(f_k)\right) = \sum_{k=1}^{2n} \alpha_k f_k.$$

Lemma 3.4. *Under above notations, we have:*
(i) If $\overline{\mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)} = \mathbb{C}^n$ *then* $\overline{\mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1})} = \widetilde{L}$.
(ii) $\Psi\left(\mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1})\right) = \mathbb{Z}f_1 + \dots + \mathbb{Z}f_{2n+1}$.

Proof. (i) Here $E(x) = \mathbb{C}^n$. By Lemma 3.3,(i), $\Phi_x : \widetilde{L} \longrightarrow \mathbb{C}^n$ is an isomorphism and by Lemma 3.3,(ii), we have $\Phi_x^{-1}(f_k(x)) = p_x(f_k)(x)$ for every $k = 1, \dots, 2n+1$, so

$$\Phi_x^{-1}(\mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)) = \mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1}).$$

Then

$$\begin{aligned}
\widetilde{L} &= \Phi_x^{-1}(\mathbb{C}^n) \\
&= \Phi_x^{-1}(\overline{\mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)}) \\
&= \overline{\mathbb{Z}\Phi_x^{-1}(f_1(x)) + \dots + \mathbb{Z}\Phi_x^{-1}(f_{2n+1}(x)))} \\
&= \overline{\mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1})}
\end{aligned}$$

(ii) Let $k_1,\dots,k_{2n+1}\in\mathbb{Z}$ and $f=k_1p_x(f_1)+\dots+k_{2n+1}p_x(f_{2n+1})$. Write $f_{2n+1}=\sum\limits_{k=1}^{2n}\alpha_kf_k$, $\alpha_1,\dots,\alpha_{2n}\in\mathbb{R}$, then

$$f=(k_1+\alpha_1k_{2n+1})p_x(f_1)+\dots+(k_{2n}+\alpha_{2n}k_{2n+1})p_x(f_{2n}),$$

so

$$\begin{aligned}\Psi(f)&=\Psi\left((k_1+\alpha_1k_{2n+1})p_x(f_1)+\dots+(k_{2n}+\alpha_{2n}k_{2n+1})p_x(f_{2n})\right)\\&=(k_1+\alpha_1k_{2n+1})f_1+\dots+(k_{2n}+\alpha_{2n}k_{2n+1})f_{2n}\\&=k_1f_1+\dots+k_{2n+1}f_{2n+1}\end{aligned}$$

Then $\Psi\left(\mathbb{Z}p_x(f_1)+\dots+\mathbb{Z}p_x(f_{2n+1})\right)\subset\mathbb{Z}f_1+\dots+\mathbb{Z}f_{2n+1}$. The same proof is used for the converse, by replacing Ψ by Ψ^{-1}. $\square$

Proposition 3.5. ([5], *Proposition 4.3*). *Let $H=\mathbb{Z}x_1+\dots+\mathbb{Z}x_p$ with $x_k\in\mathbb{R}^n$. Then H is dense in $\mathbb{R}^n$ if and only if for every $(s_1,\dots,s_p)\in\mathbb{Z}^p\backslash\{0\}$:*

$$\operatorname{rank}\begin{bmatrix}x_1&\dots&\dots&x_p\\ s_1&\dots&\dots&s_p\end{bmatrix}=\ n+1.$$

Proof of Theorem 1.2. Write $H_x=\mathbb{Z}f_1(x)+\dots+\mathbb{Z}f_{2n+1}(x)$. Since $\overline{H_x}=\mathbb{C}^n$, by Lemma 3.2, $(f_1(x),\dots,f_{2n}(x))$ is a basis of $\mathbb{C}^n$, so $f_1,\dots,f_{2n}$ are linearly independent over $\mathbb{R}$. Denote by $E=vect(f_1,\dots,f_{2n})$, then $E=\Psi(\widetilde{L})$ and it has a dimension equal to $2n$ over $\mathbb{R}$, so $\Psi:\widetilde{L}\longrightarrow E$ is an isomorphism. Since $f_{2n+1}\in vect(f_1,\dots,f_{2n})$ then by Lemma 3.4,(i), $\overline{\mathbb{Z}p_x(f_1)+\dots+\mathbb{Z}p_x(f_{2n+1})}=\widetilde{L}$. Therefore:

$$\begin{aligned}E&=\Psi(\widetilde{L})\\&=\Psi(\overline{\mathbb{Z}p_x(f_1)+\dots+\mathbb{Z}p_x(f_{2n+1})})\\&=\overline{\mathbb{Z}f(p_x(f_1))+\dots+\mathbb{Z}f(p_x(f_{2n+1}))}\\&=\overline{\mathbb{Z}f_1+\dots+\mathbb{Z}f_{2n+1}}\qquad(1)\end{aligned}$$

Let $1\leq k\leq 2n$ and $t_k\in\mathbb{R}^*$ such that $|t_k|<\frac{r_{\widetilde{G}}}{\|f_k\|}$.

• First, let's prove that $e^{t_kf_k}\in G$

Since $t_kf_k\in E$, then by (1), $t_kf_k\in\overline{\mathbb{Z}f_1+\dots+\mathbb{Z}f_{2n+1}}$. Thus there exists a sequence $(g_j)_{j\in\mathbb{N}}\subset\mathbb{Z}f_1+\dots+\mathbb{Z}f_{2n+1}$ such that $\underset{j\to+\infty}{lim}g_j=t_kf_k$. By continuity of the exponential map we have $\underset{j\to+\infty}{lim}e^{g_j}=e^{t_kf_k}$. Since $\mathbb{Z}f_1+\dots+\mathbb{Z}f_{2n+1}\subset exp^{-1}(\widetilde{G})$ then $g_j\in exp^{-1}(\widetilde{G})$, so $e^{t_kf_k}\in\widetilde{G}$, since $\widetilde{G}$ is closed in $Diff^r(\mathbb{C}^n)$.

• Second, as $|t_k|<\frac{r_G}{\|f_k\|}$, then $\|t_kf_k\|<r_{\widetilde{G}}$. Since $|t_k|\neq 0$ and $(f_1(x),\dots,f_{2n}(x))$ is a basis of $\mathbb{C}^n$, so is $(t_1f_1(x),\dots,t_{2n}f_{2n}(x))$. By the first step we conclude that $e^{t_kB_k}\in G$ for every $k=1,\dots,2n$. The proof follows then from Theorem 3.1. $\square$

The complex form of Proposition 3.5 is given in the following:

Proposition 3.6. ([5], *page 35*). *Let $H = \mathbb{Z}z_1 + \cdots + \mathbb{Z}z_p$ with $z_k \in \mathbb{C}^n$ and $z_k = \mathrm{Re}(z_k) + i\mathrm{Im}(z_k)$, $k = 1, \dots, p$. Then H is dense in $\mathbb{C}^n$ if and only if for every $(s_1, \dots, s_p) \in \mathbb{Z}^p \backslash \{0\}$:*

$$\mathrm{rank}\begin{bmatrix} \mathrm{Re}(z_1) & \dots & \dots & \mathrm{Re}(z_p) \\ \mathrm{Im}(z_1) & \dots & \dots & \mathrm{Im}(z_p) \\ s_1 & \dots & \dots & s_p \end{bmatrix} = 2n+1.$$

Proof of Corollary 1.3. The proof results directly, from Theorem 1.2 and Proposition 3.6. □

Lemma 3.7. ([1], *Corollary 1.3*). *Let G be an abelian subgroup of $GL(n, \mathbb{C})$. If G has a locally dense orbit γ in $\mathbb{C}^n$ then γ is dense in $\mathbb{C}^n$.*

Proof of Corollary 1.4. Since the matrices B_j, $1 \leq j \leq 2n+1$ commute then $\mathbb{Z}B_1 + \cdots + \mathbb{Z}B_{2n+1} \subset exp^{-1}(G)$. Hence the proof of Corollary 1.4 results directly from Corollary 1.3 and Lemma 3.7. □

Question1: *How can we characterize explicitly* $\mathrm{g} = exp^{-1}(G)$ *for any finitely generated abelian subgroup G of a lie group $\Gamma \subset Diff^r(\mathbb{C}^n)$?*

Question2: *A somewhere dense orbit of a non abelian subgroup of $Diff^r(\mathbb{C}^n)$ can always be dense in $\mathbb{C}^n$?*

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